Problems

① Prove that the ring of matrices,

$R = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a, b \in \mathbb{R} \right\}$ is a field.

Soln:

Let, $R = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a, b \in \mathbb{R} \right\}$. $(R, +, \cdot)$ is a ring with unity, the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ being the unity.

Let, $A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, $B = \begin{pmatrix} p & q \\ -q & p \end{pmatrix} \in R$. Then,

$$A \cdot B = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \cdot \begin{pmatrix} p & q \\ -q & p \end{pmatrix} = \begin{pmatrix} ap - bq & aq + bp \\ -bp - aq & -bq + ap \end{pmatrix}$$

$$B \cdot A = \begin{pmatrix} p & q \\ -q & p \end{pmatrix} \cdot \begin{pmatrix} a & b \\ -b & a \end{pmatrix} = \begin{pmatrix} ap - bq & bq + aq \\ -aq - bp & -qb + pa \end{pmatrix}.$$

Therefore, $A \cdot B = B \cdot A$ for all $A, B \in R$.

Hence $(R, +, \cdot)$ is a commutative ring with unity.

Let, $A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ be a non-zero element of S.

Then, $(a, b) \neq 0$ and $\det A = a^2 + b^2 \neq 0$.

Hence A^{-1} exists and $A^{-1} = \frac{1}{a^2+b^2} \begin{pmatrix} a & -b \\ -b & a \end{pmatrix} \in S$.

Therefore each non-zero element of the ring is a unit.

Hence $(R, +, \cdot)$ is a field.

(2) Prove that the ring of matrices, $R = \left\{ \begin{pmatrix} a & b \\ 2b & a \end{pmatrix} : a, b \in \mathbb{Q} \right\}$ is a field.

$\rightarrow$ Let, $R = \left\{ \begin{pmatrix} a & b \\ 2b & a \end{pmatrix} : a, b \in \mathbb{Q} \right\}$

$(R, +, \cdot)$ is a ring with unity, $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ being the unity.

Let, $A = \begin{pmatrix} a & b \\ 2b & a \end{pmatrix}$, $B = \begin{pmatrix} p & q \\ 2q & p \end{pmatrix} \in R$. Then,

$$A \cdot B = \begin{pmatrix} a & b \\ 2b & a \end{pmatrix} \cdot \begin{pmatrix} p & q \\ 2q & p \end{pmatrix} = \begin{pmatrix} ap + 2bq & aq + bp \\ 2bp + 2aq & 2bq + ap \end{pmatrix}$$

$$B \cdot A = \begin{pmatrix} p & q \\ 2q & p \end{pmatrix} \cdot \begin{pmatrix} a & b \\ 2b & a \end{pmatrix} = \begin{pmatrix} ap + 2bq & bp + aq \\ 2aq + 2bp & 2bq + ap \end{pmatrix}$$

Therefore,

$A \cdot B = B \cdot A$ for all $A, B \in R$.

Hence, $(R, +, \cdot)$ is a commutative ring with unity.

Let, $A = \begin{pmatrix} a & b \\ 2b & a \end{pmatrix}$ be a non zero element of R. Then $(a, b) \neq (0, 0)$ and $\det A = a^2 - 2b^2 \neq 0$ since a and b are rational.

Hence A^{-1} exists and $A^{-1} = \begin{pmatrix} a & -b \\ -2b & a \end{pmatrix} \cdot \dfrac{1}{a^2 - 2b^2} \in R$

$\therefore$ Each non zero element in R is a unit

Hence $(R, +, \cdot)$ is a field.

(3) P... matrices $\left\{ \begin{pmatrix} a & b \\ 2b & a \end{pmatrix} : a, b \in \right.$